Deep Learning Systems: Algorithms and Implementation

Convolutional Networks

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Outline

Convolutional operators in deep networks

Elements of practical convolutions

Differentiating convolutions

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Convolutional operators in deep networks

Elements of practical convolutions

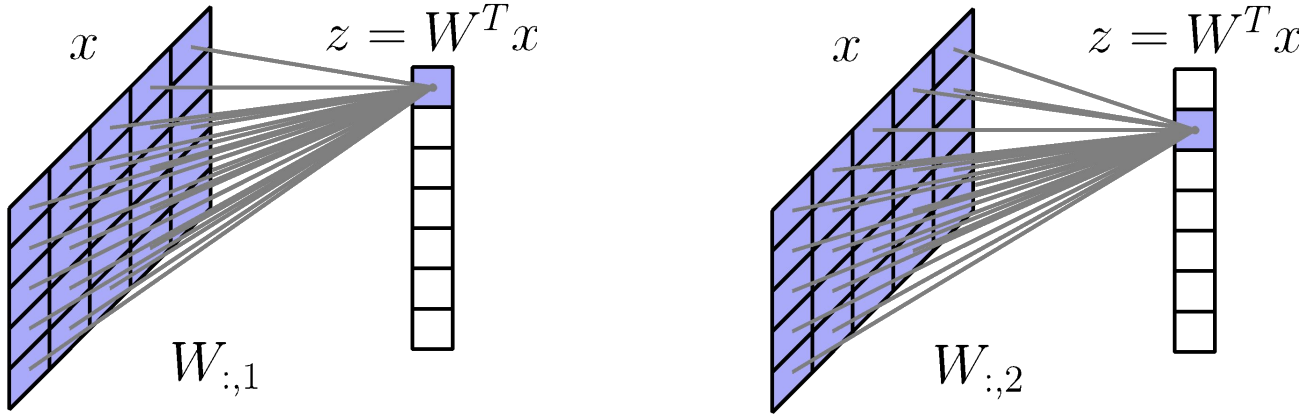
Differentiating convolutions

The problem with fully connected networks

So far we have considered networks that treat input images as vectors

This creates a substantial problem as we attempt to handle larger images: a 256x256 RGB image ⟹ ~200K dimensional input ⟹ mapping to 1000 dimensional hidden vector requires 200M parameters (for a *single layer)*

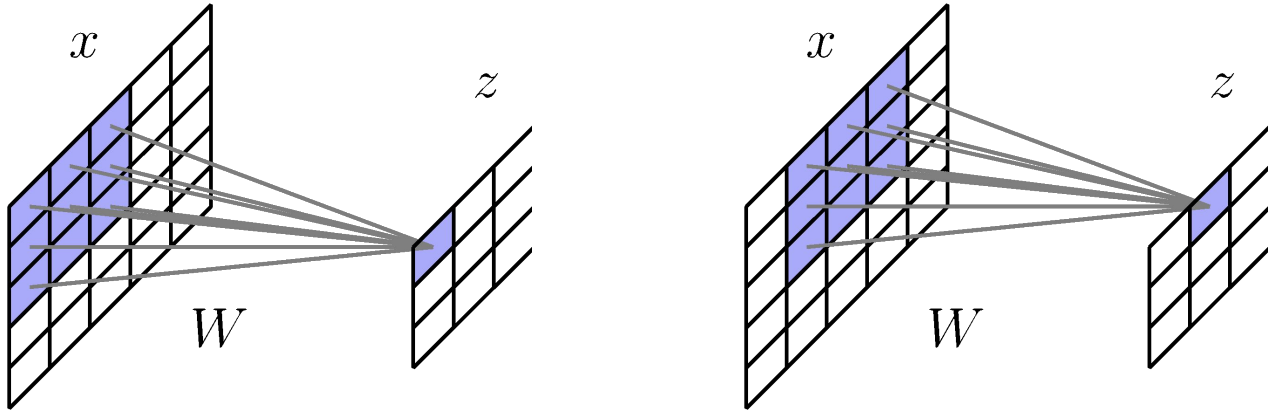
Does not capture any of the "intuitive" invariances that we expect to have in images (e.g., shifting image one pixel leads to very different next layer)



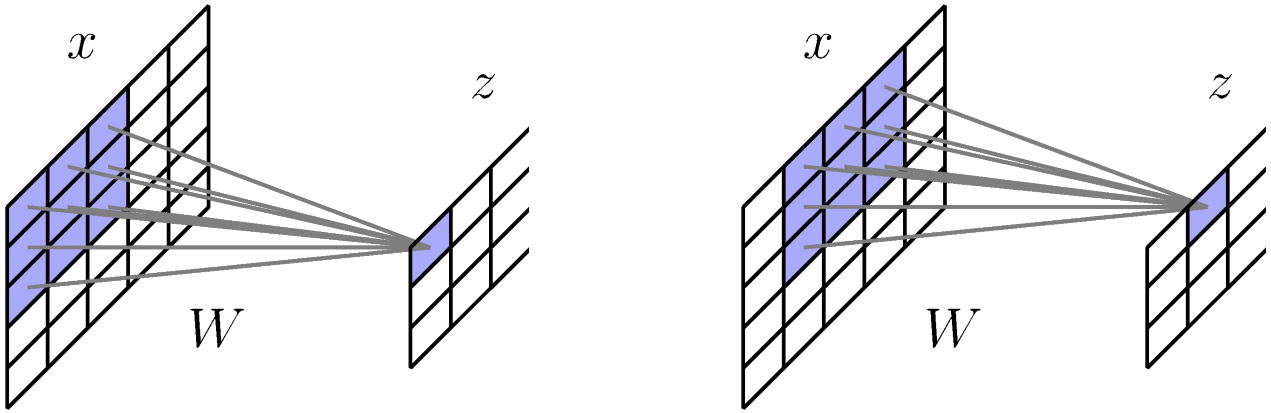
How convolutions “simplify” deep networks

Convolutions combine two ideas that are well-suited to processing images

1. Require that activations between layers occur only in a “local” manner, and treat hidden layers themselves as spatial images
2. Share weights across all spatial locations



Advantages of convolutions



Drastically reduces the parameter count

- 256x256 grayscale image $\Longrightarrow$ 256x256 single-channel hidden layer: 4 *billion parameters* in fully connected network to *9 parameters* in 3x3 convolution

Captures (some) “natural” invariances

- Shifting input image one pixel to the right shifts creates a hidden shifts the hidden unit “image”

Convolutions in detail

Convolutions are a basic primitive in many computer vision and image processing algorithms

Idea is to "slide" the weights $k \times k$ weight w (called a filter, with kernel size k) over the image to produce a new image, written $y = z * w$

z_{11}	z_{12}	z_{13}	z_{14}	z_{15}
z_{21}	z_{22}	z_{23}	z_{24}	z_{25}
z_{31}	z_{32}	z_{33}	z_{34}	z_{35}
z_{41}	z_{42}	z_{43}	z_{44}	z_{45}
z_{51}	z_{52}	z_{53}	z_{54}	z_{55}

$*$

w_{11}	w_{12}	w_{13}
w_{21}	w_{22}	w_{23}
w_{31}	w_{32}	w_{33}

$=$

y_{11}	y_{12}	y_{13}
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w_{21}	w_{22}	w_{23}
w_{31}	w_{32}	w_{33}

$=$

y_{11}	y_{12}	y_{13}
y_{21}	y_{22}	y_{23}
y_{31}	y_{32}	33

$$y_{11} = z_{11}w_{11} + z_{12}w_{12} + z_{13}w_{13} + z_{21}w_{21} + \ldots$$

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w_{31}	w_{32}	w_{33}

$=$

y_{11}	y_{12}	y_{13}
y_{21}	y_{22}	y_{23}
y_{31}	y_{32}	33

$$y_{12} = z_{12}w_{11} + z_{13}w_{12} + z_{14}w_{13} + z_{22}w_{21} + \dots$$

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w_{31}	w_{32}	w_{33}

$=$

y_{11}	y_{12}	y_{13}
y_{21}	y_{22}	y_{23}
y_{31}	y_{32}	33

$$y_{13} = z_{13}w_{11} + z_{14}w_{12} + z_{15}w_{13} + z_{23}w_{21} + \ldots$$

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y_{11}	y_{12}	y_{13}
y_{21}	y_{22}	y_{23}
y_{31}	y_{32}	33

$$y_{21} = z_{21}w_{11} + z_{22}w_{12} + z_{23}w_{13} + z_{31}w_{21} + \ldots$$

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y_{11}	y_{12}	y_{13}
y_{21}	y_{22}	y_{23}
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$$y_{22} = z_{22}w_{11} + z_{23}w_{12} + z_{24}w_{13} + z_{32}w_{21} + \ldots$$

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y_{11}	y_{12}	y_{13}
y_{21}	y_{22}	y_{23}
y_{31}	y_{32}	33

$$y_{23} = z_{23}w_{11} + z_{24}w_{12} + z_{25}w_{13} + z_{33}w_{21} + \ldots$$

Convolutions in image processing

Convolutions (typically with *prespecified* filters) are a common operation in many computer vision applications: convolution networks just move to *learned* filters

Original image z

Gaussian blur

$$z * \begin{bmatrix} 1 & 4 & 7 & 4 & 1 \\ 4 & 16 & 26 & 16 & 4 \\ 7 & 26 & 41 & 26 & 7 \\ 4 & 16 & 26 & 16 & 4 \\ 1 & 4 & 4 & 4 & 1 \end{bmatrix} /273$$

Image gradient

$$\left(\left(z * \begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix} \right)^2 + \left(z * \begin{bmatrix} -1 & -2 & -1 \\ 0 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix} \right)^2 \right)^{\frac{1}{2}}$$

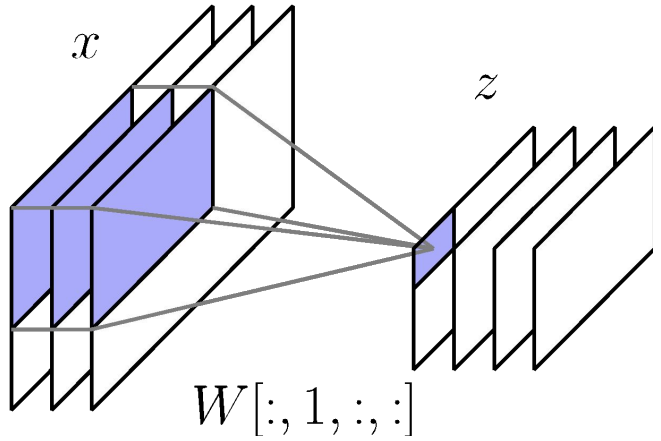
Convolutions in deep networks

Convolutions in deep networks are virtually always *multi-channel* convolutions: map multi-channel (e.g., RGB) inputs to multi-channel hidden units

- $x \in \mathbb{R}^{h \times w \times c_{in}}$ denotes c_{in} channel, size $h \times w$ image input
- $z \in \mathbb{R}^{h \times w \times c_{out}}$ denotes c_{out} channel, size $h \times w$ image input
- $W \in \mathbb{R}^{c_{in} \times c_{out} \times k \times k}$ (order 4 tensor) denotes convolutional filter

Multi-channel convolutions contain a convolutional filter for *each input-output channel pair*, single output channel is sum of convolutions over all input channels

$$z[:,:,s] = \sum_{r=1}^{c_{in}} x[:,:,r] * W[r,s,:,:]$$



Multi-channel convolutions in matrix-vector form

There is, in my view, a more intuitive way to think about multi-channel convolutions: they are a generalization of traditional convolutions with scalar multiplications replaced by matrix-vector products

These are each $\mathbb{R}^{c_{out} \times c_{in}}$ matrices

These are each vectors in $\mathbb{R}^{c_{in}}$

$$\begin{array}{|c|c|c|c|c|} \hline x_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ \hline x_{21} & x_{22} & x_{23} & x_{24} & x_{25} \\ \hline x_{31} & x_{32} & x_{33} & x_{34} & x_{35} \\ \hline x_{41} & x_{42} & x_{43} & x_{44} & x_{45} \\ \hline x_{51} & x_{52} & x_{53} & x_{54} & x_{55} \\ \hline \end{array} * \begin{array}{|c|c|c|} \hline W_{11} & W_{12} & W_{13} \\ \hline W_{21} & W_{22} & W_{23} \\ \hline W_{31} & W_{32} & W_{33} \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline z_{11} & z_{12} & z_{13} \\ \hline z_{21} & z_{22} & z_{23} \\ \hline z_{31} & z_{32} & z_{33} \\ \hline \end{array}$$

$$z_{22} = W_{11}x_{22} + W_{12}x_{23} + W_{13}x_{24} + W_{21}x_{32} + \dots$$

These are matrix-vector products

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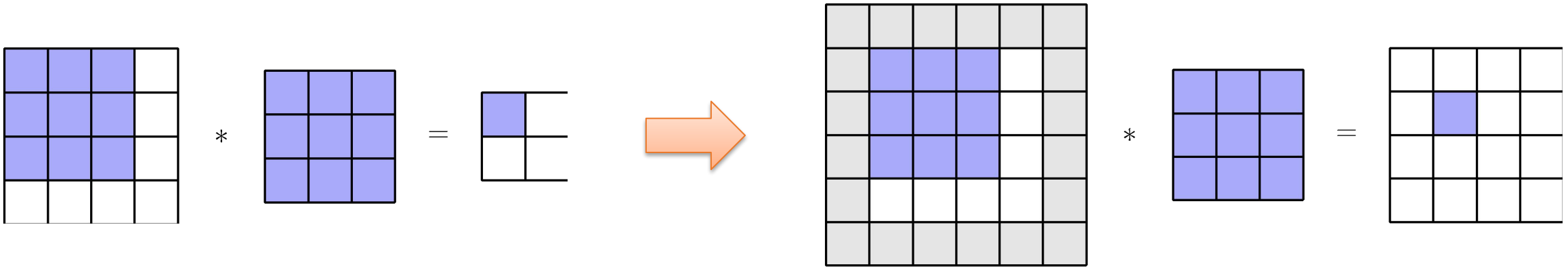
Padding

Challenge: “Naïve” convolutions produce a smaller output than input image

Solution: for (odd) kernel size k, pad input with $(k-1)/2$ zeros on all sides, results in an output that is the same size as the input

- Variants like circular padding, padding with mean values, etc

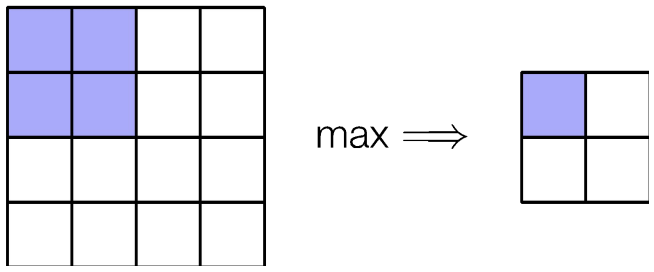


Strided Convolutions / Pooling

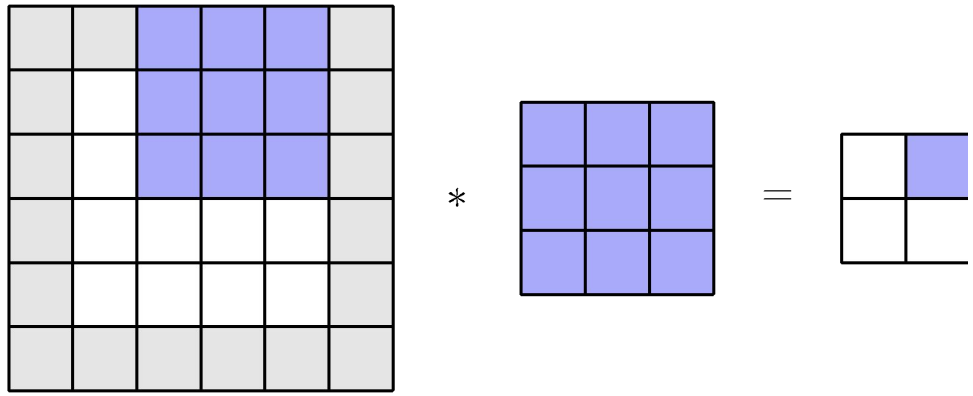
Challenge: Convolutions keep the same resolution of the input at each layer, don't naively allow for representations at different "resolutions"

Solution #1: incorporate max or average *pooling* layers to aggregate information

Solution #2: slide convolutional filter over image in increments >1 (= stride)



Max pooling

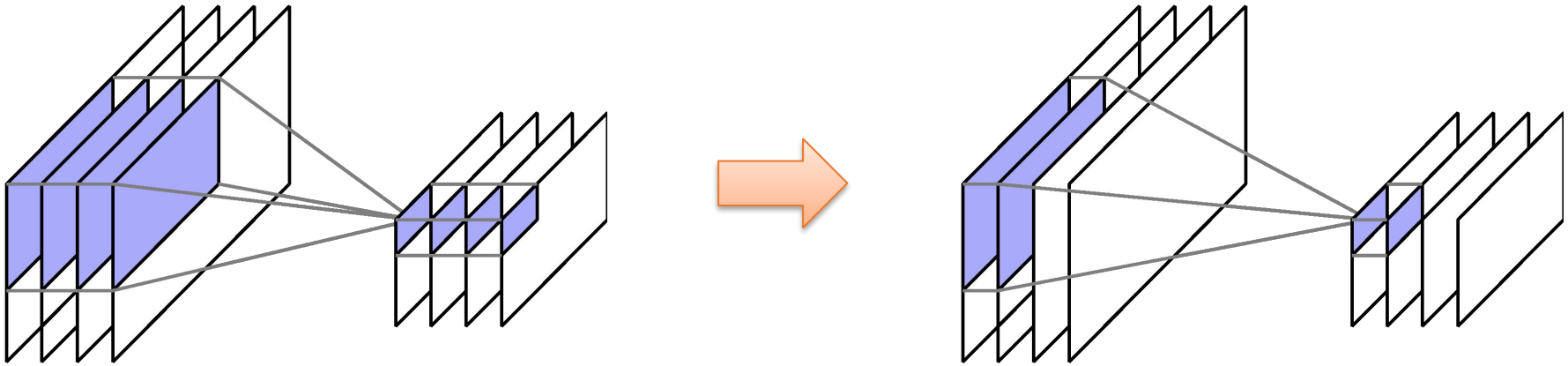


Strided convolution

Grouped Convolutions

Challenge: for large numbers of input/output channels, filters can still have a large number of weights, can lead to overfitting + slow computation

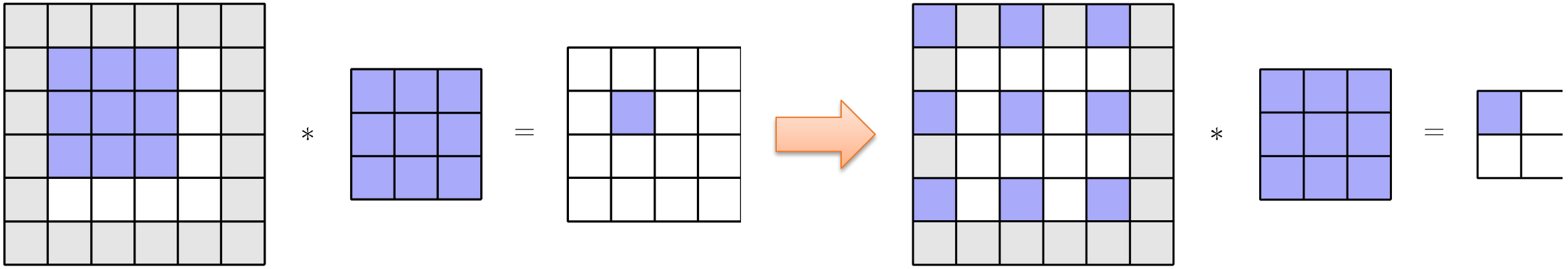
Solution: Group together channels, so that groups of channels in output only depend on corresponding groups of channels in input (equivalently, enforce filter weight matrices to be block-diagonal)



Dilations

Challenge: Convolutions each have a relatively small receptive field size

Solution: *Dilate* (spread out) convolution filter, so that it covers more of the image (see also: later architectures we will discuss, like self-attention layers); note that getting an image of the same size again requires adding more padding



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Differentiating convolutions

What is needed to differentiate convolution?

Recall that in order to integrate any operation into a deep network, we need to be able to multiply by its partial derivatives (adjoint operation)

So if we define our operation

$$z = \mathrm{conv}(x, W)$$

how do we multiply by the adjoints

$$\bar{v}\frac{\partial \mathrm{conv}(x, W)}{\partial W}, \qquad \bar{v}\frac{\partial \mathrm{conv}(x, W)}{\partial x}$$

Refresher on differentiating matrix multiplication

Let's consider the simpler case of a matrix-vector product operation

$$z = Wx$$

Then $\frac{\partial z}{\partial x} = W$, so we need to compute the adjoint product

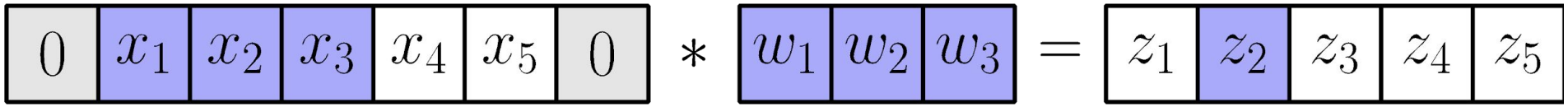
$$\bar{v}^T W \iff W^T \bar{v}$$

In other words, for a matrix vector multiply operation Wx, computing the backwards pass requires multiplying by the *transpose* W^T

So what is the "transpose" of a convolution?

Convolutions as matrix multiplication: Version 1

To answer this question, consider a 1D convolution to keep things a bit simpler:



We can write a 1D convolution $x * w$ (e.g., with zero padding) as a matrix multiplication $\widehat{W}x$ for some $\widehat{W}$ properly defined in terms of the filter w

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{bmatrix} = x * w = \underbrace{\begin{bmatrix} w_2 & w_3 & 0 & 0 & 0 \\ w_1 & w_2 & w_3 & 0 & 0 \\ 0 & w_1 & w_2 & w_3 & 0 \\ 0 & 0 & w_1 & w_2 & w_3 \\ 0 & 0 & 0 & w_1 & w_2 \end{bmatrix}}_{\widehat{W}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

The adjoint of a convolution

So how can we multiply by the transpose $\widehat{W}^T$?

Written out as in the previous slide, it's quite easy:

$$\widehat{W}^T = \begin{bmatrix} w_2 & w_1 & 0 & 0 & 0 \\ w_3 & w_2 & w_1 & 0 & 0 \\ 0 & w_3 & w_2 & w_1 & 0 \\ 0 & 0 & w_3 & w_2 & w_1 \\ 0 & 0 & 0 & w_3 & w_2 \end{bmatrix}$$

But notice that the operation $\widehat{W}^T v$ it itself just a convolution with the "flipped" filter: $\begin{bmatrix} w_3 & w_2 & w_1 \end{bmatrix}$; $\Longrightarrow$ adjoint operator $\bar{v}\frac{\partial \text{conv}(x,W)}{\partial x}$ just requires convolving $\bar{v}$ with a the flipped W!

Convolutions as matrix multiplication: Version 2

What about the other adjoint, $\bar{v}\frac{\partial \text{conv}(x,W)}{\partial W}$?

For this term, observe that we can *also* write the convolution as a matrix-vector product treating the *filter* as the vector

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{bmatrix} = x * w = \begin{bmatrix} 0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \\ x_3 & x_4 & x_5 \\ x_4 & x_5 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

So adjoint requires multiplying by the transpose of this x-based matrix (actually a relatively practical approach, see future lecture on the "im2col" operation)